UNIVERSAL MODELS VIA EMBEDDING AND REDUCTION FOR LOCALLY CONFORMAL SYMPLECTIC STRUCTURES

JUAN C. MARRERO, DAVID MARTÍNEZ TORRES, AND EDITH PADRÓN

ABSTRACT. We obtain universal models for several types of locally conformal symplectic manifolds via pullback or reduction. The relation with recent embedding results for locally conformal Kähler manifolds is discussed.

1. INTRODUCTION AND STATEMENT OF RESULTS

A manifold M endowed with a nondegenerate 2-form Φ is an almost symplectic manifold. An almost symplectic manifold (M, Φ) is said to be *locally conformal symplectic (l.c.s.)* if for each $x \in M$, there exist an open neighborhood U of x and a function $\sigma : U \to \mathbb{R}$ such that $(U, e^{-\sigma}\Phi)$ is a symplectic manifold, i.e., $d(e^{-\sigma}\Phi) = 0$ ([11, 34]). This type of manifolds are included in the category of Jacobi manifolds. In fact, the leaves of the characteristic foliation of a Jacobi manifold are contact or l.c.s. manifolds (see, for instance [6, 11, 16]). For manifolds of dimension greater than 2, an assumption we make from now on, the l.c.s. condition is equivalent to

$$d\Phi = \omega \wedge \Phi, \tag{1}$$

where ω is a closed 1-form, the *Lee 1-form.* The 2-form Φ is referred to as a *l.c.s. form.* Recalling that any closed 1-form defines a twisted de Rham cohomology, equation 1 describes a l.c.s. form as a non-degenerate 2-form which is closed in a twisted de Rham cohomology complex. This viewpoint is relevant to draw analogies with symplectic geometry.

If Φ is a l.c.s. form, then so is $f\Phi$ for any $f \in C^\infty(M)$ no-where vanishing. The l.c.s. forms Φ and $f\Phi$ are said to belong to the same *conformal class.* We will always assume f to be positive, so our conformal classes will be -strictly speaking-positive conformal classes.

A salient feature of l.c.s. structures is that they provide a framework for Hamiltonian mechanics more general than the one provided by symplectic structures (see [34] or for instance, the recent paper by Marle [20] where the theory of conformally Hamiltonian vector fields was applied to the Kepler problem).

It is natural to investigate up to which extent properties of symplectic manifolds and techniques in symplectic geometry generalize to l.c.s. geometry. In the symplectic context, for instance, there is a noteworthy work on embeddings (see, for example, [33]) and on reduction (see the book by Ortega and Ratiu [30] and references therein; see also the book by Marsden et al [21] for Hamiltonian reduction by stages).

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For l.c.s. geometry, some results on the group of automorphisms of a l.c.s. structure [13], on reduction [14], on Moser stability type results [1] and on existence of l.c.s. structures on open manifolds via a h-principle [7] have been obtained. Another very active line of research in the subject is centered in *locally conformal Kähler (l.c.K.)* manifolds. These are complex manifolds with a Hermitian metric locally conformal to a Kähler one; the underlying l.c.s. structure is defined by the 2-form associated to the Hermitian metric. The role played by l.c.K manifolds within l.c.s. manifolds is analogous to the one of Kähler manifolds within symplectic ones. Among the very remarkable recent results in l.c.K. geometry, one finds an analog of Kodaira embedding theorem for a subclass of l.c.K manifolds [27].

Kodaira embedding theorem is a very good example of a result in Kähler geometry, which with the appropriate formulation holds also in symplectic geometry. Namely, Tischler [33] proves that any integral, compact, symplectic manifold symplectically embeds in some projective space. In other words, projective spaces with the integral Fubini-Study symplectic form are universal models for integral symplectic structures in compact manifolds. With regard to in which sense Tischler embedding relates to Kodaira's result, it is known that in general one cannot find holomorphic embeddings which at the same time pull back the Fubini-Study metric to a (suitable multiple) of the given Kähler metric. But one easily goes from the holomorphic to the symplectic embedding by applying Moser stability to the convex combination of the two cohomologous Kähler forms.

Motivated by the aforementioned results of Tischler, and Ornea and Verbitsky, in this paper we take up the problem of investigating the existence of compact universal models for l.c.s. structures. Roughly, this amounts to finding families of compact l.c.s. manifolds -which will be rather special- together with a procedure -either pullback or reduction (though for the latter compactness will be dropped)- which allows us to produce any given l.c.s. structure under reasonable constraints.

Our first result provides a positive answer for a type of l.c.s. structures, exact l.c.s. structures with integral period lattice on compact manifolds (see sections 3 and 4 for background on l.c.s. structures).

Theorem 1. *Let (M, Φ, α) be a compact manifold of dimension $2n$ endowed with an exact l.c.s. structure, whose Lee form ω has integral period lattice. Then, for any $N \geq 4n+2$, there exist an embedding $\Psi\colon M \to S^{2N-1} \times S^1$ and a real number $c, c > 0$, such that*

$$\Psi^*(c\eta_N) = \alpha, \quad \Psi^*(d\theta) = \omega, \quad \Psi^*(c\Phi_N) = \Phi, \tag{2}$$

where η_N is the standard contact 1-form on S^{2N-1}, $d\theta$ the standard integral 1-form on the circle and Φ_N the associated standard l.c.s. structure with integral period lattice on $S^{2N-1} \times S^1$.

Using the language introduced in section 3, the embedding Ψ is a full strict morphism into $(S^{2N-1} \times S^1, d\theta)$ which pulls back the homothety class of η_N into the homothety class of α (and thus does the same for the l.c.s. forms).

As a consequence of Theorem 1, we deduce the following result:

Corollary 1. *Let M be a compact manifold of dimension $2n$ endowed with a l.c.s. structure, whose Lee form ω is not zero in some point of M, it has integral period lattice and it is parallel with respect to a Riemannian metric on M. Then, for any $N \geq 4n+2$, there exist an embedding $\Psi\colon M \to S^{2N-1} \times S^1$ and a real number $c, c > 0$, such that*

$$\Psi^*(c\eta_N) = \alpha, \quad \Psi^*(d\theta) = \omega, \quad \Psi^*(c\Phi_N) = \Phi,$$

where η_N is the standard contact 1*-form on S^{2N-1}, $d\theta$ the standard integral 1-form on the circle and Φ_N the associated standard l.c.s. structure with integral period lattice on $S^{2N-1} \times S^1$.*

The manifold $S^{2N-1} \times S^1$ admits many l.c.K. structures with integral period lattice associated to diffeomorphisms with linear Hopf manifolds H_A [15] (see section 5 for background on l.c.K. structures). The standard l.c.s. form with integral period lattice in theorem 1 underlies the l.c.K. form associated to obvious diffeomorphisms to several diagonal Hopf manifolds. In [26, 27] it is shown that any compact l.c.K. manifold of complex dimension at least 3 with automorphic potential -an appropriate generalization of exact l.c.s. structures in the l.c.K. setting for which the underlying l.c.s. structure is exact- admits a holomorphic embedding into a linear Hopf manifold H_A.

Our second result asserts that the relation between theorem 1 and Ornea and Verbitsky embedding, mimics the relation between Tischler and Kodaira embeddings.

Theorem 2. *Let (J, Φ_g, r) be a l.c.K. structure with automorphic potential and integral period lattice on a compact manifold M. Let $(M, J, \Phi_{g'}, r')$ be the l.c.K. structure with automorphic potential and integral period lattice induced by any of the holomorphic embeddings $\Psi\colon (M, J) \to (H_A, J_A)$ in* [26, 27]*, where (H_A, J_A) is endowed with a l.c.K. structure Φ_A with integral period lattice as described in* [15, 28]*. Then there exist diffeomorphisms $\varphi\colon M \to M$ and $\phi\colon H_A \to S^{2N-1} \times S^1$, such that*

- *ϕ pulls back the standard l.c.s. form Φ_N on the sphere $S^{2N-1} \times S^1$ to the positive conformal class of Φ_A.*
- *φ is isotopic to the identity and pulls back the l.c.s. form $\Phi_{g'}$ to the positive conformal class of Φ_g.*

Therefore,

$$(\phi \circ \Psi \circ \varphi)^* \Phi_N = f \Phi_g,$$

where f is a strictly positive function. Equivalently, $\phi \circ \Psi \circ \varphi$ is a full morphism which pulls back the conformal twisted cohomology class of Φ_N into the conformal twisted cohomology class of Φ_g.

The diffeomorphisms φ and ϕ are constructed via the Moser stability result in [1].

Theorem 1 provides a way of producing all exact l.c.s. structures with integral period lattice on compact manifolds via pullback (or restriction). Very much as in symplectic geometry, one can give conditions so that a reduction process is possible for l.c.s. structures [14]. Thus one may ask about the existence of universal models for l.c.s. structures via reduction. Our third main theorem gives a positive answer to this question for l.c.s. structures of the first kind on manifolds of finite type, and it is a natural generalization of results in [10, 18, 19].

Theorem 3. *Let (M, Φ, α) be a finite type manifold of dimension $2n$, endowed with a l.c.s. structure of the first kind with rank k period lattice Λ. Then, for any $N \geq 4n + k$, the l.c.s. manifold (M, Φ, α) is isomorphic to the l.c.s. reduction of certain strongly reducible submanifold of*

$$(\mathbb{R} \times \mathcal{J}^1(\mathbb{T}^k \times \mathbb{R}^N), \Phi_{N,\Lambda}, \alpha_{N,k}, \omega_\Lambda).$$

The l.c.s. structure $\Phi_{N,\Lambda}$ is of the first kind with potential 1-form $\alpha_{N,k}$ the canonical 1-form in the first jet space of $\mathbb{T}^k \times \mathbb{R}^N$. Its Lee form ω_Λ has period lattice Λ.

We also prove an equivariant version of the previous theorem.

Theorem 4. *Let G be a compact connected Lie group which acts on a finite type l.c.s. manifold (M, Φ, α) of the first kind with rank k period lattice Λ, via a l.c.s. action $\psi : G \times M \to M$ of the first kind. Then, for a sufficiently large integer N, (M, Φ, α, ψ) is isomorphic to the l.c.s. equivariant reduction by a certain G-invariant strongly reducible submanifold of the l.c.s. structure of the first kind $(M_{k,N} = \mathbb{R} \times \mathcal{J}^1(\mathbb{T}^k \times \mathbb{R}^N), \Phi_{N,\Lambda}, \alpha_{k,N}, \omega_\Lambda, \psi_{k,N})$, where $\psi_{k,N} : G \times M_{k,N} \to M_{k,N}$ is a l.c.s. action of the first kind.*

In looking at the problem of existence of universal l.c.s. manifolds linking with the results in [26, 27], one is naturally led to ask about the existence of (compact) universal models for compact manifolds endowed with an arbitrary 1-form. This is a problem that was addressed in much more generality in [23], where universal models for (principal) connections on principal bundles for compact groups were constructed. For $U(1)$ the universal models are $S^{2N-1} \to \mathbb{CP}^{N-1}$ with the standard contact 1-form η_N. Any 1-form on a manifold M defines a connection on the trivial principal bundle $M \times U(1)$. By [23] one produces a bundle morphism $M^n \times U(1) \to S^{8n+3}$, which composed by the right with the inclusion $M \to M \times \{1\}$ pulls back the standard contact 1-form to the given 1-form. It turns out that if one is interested not in every compact group but just in $U(1)$, a slightly different proof allows to cut down substantially the dimension of the target sphere from $8n + 3$ to $4n + 3$.

Theorem 5. *Let M be a compact manifold of dimension n and Θ be a 1-form on M. Then for any $N \geq 2n + 2$, there exist an embedding $\Psi : M \to S^{2N-1}$ and a real number $c, c > 0$, such that*

$$\Psi^*(c\eta_N) = \Theta.$$

In particular, if Θ is a contact 1-form one obtains an strict contact embedding between the contact manifold (M, Θ) and $(S^{2N-1}, c\eta_N)$.

The paper is organized as follows. In section 2 we will show that a universal model (via embeddings) of a compact manifold endowed with a 1-form is the $(2N - 1)$-sphere with its standard contact structure (up to the multiplication by a constant). In section 3 we will recall some aspects of twisted de Rham complexes and their cohomology; this setting allows to introduce l.c.s. structures as a twisted version of symplectic structures. In section 4 will prove that for a compact exact l.c.s. manifold M with integral period lattice, there exist a natural number N and an embedding which pulls back the standard l.c.s. structure with integral period lattice in $S^{2N-1} \times S^1$ to the l.c.s structure on M. In the particular case of l.c.K. manifolds, we will relate our results with the ones proved recently by Ornea and Verbitsky. In section 6, we will describe a universal model for reduction of a l.c.s. manifold of the first kind (theorem 3). An equivariant version of this last result is proved in section 7 (see theorem 4). The paper ends with our conclusions, a description of future research directions and an appendix where we show the non-exactness of the Oeljeklaus-Toma l.c.K. structures.

2. Universal models for 1-forms

In this section we show that the spheres with their standard contact structures are compact universal models for compact manifolds endowed with 1-forms. An upper bound for the dimension of the corresponding model sphere in terms of the dimension of the given manifold is also obtained.

Given a manifold endowed with a 1-form (M, Θ), it is always possible to induce the 1-form via an embedding in some Euclidean space endowed with a linear 1-form. Specifically, in $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ with coordinates $x_1, y_1, \ldots, x_n, y_n$, we consider

the Liouville 1-form

$$\lambda_n = \sum_{j=1}^{n} y_j dx_j.$$

A manifold M can always be embedded as a closed submanifold of some Euclidean space $\mathbb{R}^N$, and Θ can be assumed to be the restriction of $\bar{\Theta} \in \Omega^1(\mathbb{R}^N)$. Using the universal property of the Liouville 1-form in the cotangent bundle, the restriction of $\bar{\Theta} \colon \mathbb{R}^N \to T^*\mathbb{R}^N$ is shown to provide an embedding with the desired property.

If our manifold is compact, we would like to have a similar result but with compact universal models as well. Work of Narasimhan and Ramanan [23] shows that a solution is given by (S^{2n-1}, η_n), where the standard (contact) 1-form η_n is the restriction to the sphere of the 1-form

$$\eta_n = \frac{1}{2} \sum_{j=1}^{n} (y_j dx_j - x_j dy_j).$$

Their result fits into the more general framework of existence of universal connections for principal bundles for compact groups. More precisely, they give a common construction for all unitary groups which includes a bound in the dimension of the target sphere. If one is just interested in $U(1)$, it is possible to find an approach which allows to obtain target spheres of smaller dimension than in [23].

Proof of theorem 5. Firstly, Whitney's Theorem grants the existence of an embedding $i \colon M \to \mathbb{R}^{2n}$. We let U be a neighborhood of $i(M)$ such that its closure $\bar{U}$ is compact. Denote by $\bar{\Theta}$ an extension of Θ to $\bar{U}$. Then,

$$\bar{\Theta} = \sum_{i=1}^{p} f_i dx_i, \text{ with } p \leq 2n, \tag{3}$$

where $(x_1, \ldots, x_{2n})$ are the restriction to $\bar{U}$ of the standard coordinates in $\mathbb{R}^{2n}$, and $f_i \in C^\infty(\bar{U})$.

Since $\bar{U}$ is compact, there exists $r_1 > 0$ such that

$$\sum_{k=1}^{p} ((f_k(x))^2 + (x_k)^2) < r_1^2, \ \ \forall x = (x_1, \ldots, x_{2n}) \in U.$$

Now, we consider the map $\Psi_1 : U \to \mathbb{R}^{2p+2}$ given by

$$\Psi_1(x) = (x_1, f_1(x), \ldots, x_p, f_p(x), \sqrt{r_1^2 - \sum_{k=1}^{p} ((f_k(x))^2 + (x_k)^2)}, 0)$$

which satisfies that $\Psi_1(M) \subseteq S^{2p+1}(r_1)$ and $\Psi_1^*(\eta_{p+1}) = \bar{\Theta} - d\varphi$, where $S^{2p+1}(r_1)$ is the sphere of dimension $2p+1$ and radius r_1, and φ is the function on U given by

$$\varphi = \frac{1}{2} (\sum_{k=1}^{p} f_k x_k).$$

Using again that $\bar{U}$ is compact, we deduce that there exists $r_2 > 0$ such that

$$\gamma(x) = 1 + (\varphi(x))^2 + \sum_{k=1}^{p} ((f_k(x))^2 + (x_k)^2) < r_2^2, \text{ for all } x \in U.$$

Then, the function $\Psi_2 \colon U \to \mathbb{R}^{2p+4}$ defined by

$$\Psi_2(x) = (x_1, f_1(x), \ldots, x_p, f_p(x), \sqrt{r_2^2 - \gamma(x)}, 0, \varphi(x), 1)$$

induces an embedding $\Psi_2 : U \to S^{2p+3}(r_2)$ such that

$$\Psi_2^*(\eta_{p+2}) = \bar{\Theta}.$$

Finally, if we consider the homothety $\Psi_3 : S^{2p+3}(r_2) \to S^{2p+3} = S^{2p+3}(1)$ given by

$$\Psi_3(x) = \frac{x}{r_2},$$

we have that $\Psi_3^*(\eta_{p+2}) = \frac{1}{r_2^2}\eta_{p+2}$. This ends the proof of our result. □

Remark 1. There is a clear analogy between the proof of theorem 5 and Tischler embedding theorem: as a first step one obtains a map into the sphere (resp. projective space) using basically the universal property of cotangent bundles (resp. that $\mathbb{CP}^\infty$ is the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$). That map gives a solution up to an exact 1-form (resp. 2-form). Then one needs to use special properties of the standard 1-forms η_N (resp. the Fubini-Study 2-forms) which makes a correction possible at the expense of increasing by two the dimension of the target.

Remark 2. Our proof is similar to the lemma in [23] section 3, which allows to obtain universal models for principal connections on trivial $U(n)$-bundles over (subsets of) Euclidean space. The difference is that what we make in two steps (firstly getting the result up to an exact form and then finding a suitable correction) in [23] is done in just one step and for all unitary groups. It is that what allows to cut down the dimension from $8n + 3$ to m, with $m \leq 4n + 3$.

3. Twisted de Rham complexes and local conformal closedness

In this section we recall a few facts about twisted de Rham differentials and their cohomology, which will be useful for our understanding of l.c.s. structures.

Let M be a manifold. The vector space of smooth functions acts on $\Omega^*(M)$ by $C^\infty(M)$-automorphisms

$$\Omega^*(M) \xrightarrow{e^f} \Omega^*(M),\ f \in C^\infty(M).$$

This is not a chain map but it becomes so if we consider the complexes

$$\Omega^*(M, d) \xrightarrow{e^f} \Omega^*(M, d_{df}),\ f \in C^\infty(M), \tag{4}$$

where we use the twisted de Rham differential

$$d_{df}(\alpha) := d\alpha - df \wedge \alpha. \tag{5}$$

Any 1-form ω can be used to twist the de Rham differential into d_ω as in (5) substituting df by ω. In this case $d_\omega^2 = 0$ if and only if ω is closed. Generalizing (4), smooth functions act on twisted de Rham complexes

$$\Omega^*(M, d_\omega) \xrightarrow{e^f} \Omega^*(M, d_{\omega+df}),\ f \in C^\infty(M), \tag{6}$$

and the isotropy of any twisted de Rham complex is determined by the constant functions. We call the equivalence classes *conformal classes of twisted de Rham complexes*; we speak of *homothety classes of twisted de Rham complexes* if we just consider the action of constant functions. Clearly, twisted de Rham complexes are in bijection with closed 1-forms; the action of functions described above corresponds to the action given by adding the differential of the function, and conformal classes of twisted de Rham complexes correspond to cohomology classes of 1-forms. In particular the conformal class of de Rham complex corresponds to exact 1-forms.

If ω is closed, the cohomology of the complex $\Omega^*(M, d_\omega)$ is the *twisted de Rham cohomology* $H^*_\omega(M)$ (also referred to in the literature as *Lichnerowicz cohomology* or *Morse-Novikov cohomology*), and (6) induces isomorphisms of twisted cohomologies. The twisted de Rham cohomology of a conformal class of twisted de Rham complexes is the twisted de Rham cohomology of any of its representatives. The

homotethy class of the twisted de Rham cohomology of a twisted complex is its twisted de Rham cohomology modulo automorphisms induced by the constants.

Let ω be a closed 1-form in M. Then ω can be identified with the additive character

$$\omega\colon H_1(M,\mathbb{Z}) \to \mathbb{R}.$$

The image of $H_1(M,\mathbb{Z})$ (or $\pi_1(M)$) by ω is a lattice Λ inside of $\mathbb{R}$. We define the *period lattice and rank* of (M,ω) to be Λ and its rank, respectively. In particular a discrete period lattice is the same as a rank 1 period lattice. We will say that ω has *integral period lattice* if $\Lambda = \mathbb{Z} \subset \mathbb{R}$. These are invariants of the conformal classes of twisted de Rham complexes.

Let (M,ω) and (M',ω') be manifolds endowed with closed 1-forms. A smooth map $\phi\colon (M,\omega) \to (M',\omega')$ is a *morphism* if it pullbacks the conformal class of $\Omega^*(M,d_{\omega'})$ into the conformal class of $\Omega^*(M,d_\omega)$. The morphism is *strict* if it maps one twisted complex into the other. Alternatively, ϕ is a morphism if $[\phi^*\omega'] = [\omega] \in H^1_{\mathrm{dR}}(M)$, and it is strict if the equality occurs at the level of 1-forms. If $\phi^*\omega' = \omega + df$ we call f a *scaling function* (which is unique up to constants).

For a morphism $\phi\colon (M,\omega) \to (M',\omega')$ we have $\Lambda \subset \Lambda'$, and thus it is rank decreasing. A morphism is called *full* if $\Lambda = \Lambda'$.

Given a morphism $\phi\colon (M,\omega) \to (M',\omega')$ and f a scaling function, there is an induced homomorphism

$$\begin{aligned} \phi^*\colon H^*_{\omega'}(M') &\longrightarrow H^*_\omega(M) \\ \beta' &\longmapsto \mathrm{e}^{-f}\phi^*\beta'. \end{aligned} \tag{7}$$

To get rid of the choice of scaling function one has to pass to the homothety class of the twisted de Rham complexes.

3.1. **Twisted de Rham cohomology and de Rham cohomology.** There are two natural ways in which twisted de Rham cohomology can be related to de Rham cohomology. They correspond to ways of neglecting the non-exactness of ω: working locally or going to a suitable covering space.

3.1.1. *Local conformal closedness.* Recall that a form $\beta \in \Omega^k(M)$ is said to be *locally conformally closed* if for each $x \in M$, there exist an open neighborhood U of x and a function $\sigma : U \to \mathbb{R}$ such that $e^{-\sigma}\beta$ is closed.

Remark 3. Depending on the local behaviour of β there might be no uniqueness up to additive constant in the choice of σ. One way to attain such uniqueness is to ask β at each point not to have isotropic hyperplanes.

Let U_i, $i \in I$, be an open cover so that $\omega_{|U_i}$ is exact. Then the inclusion $(U_i,0) \hookrightarrow (M,\omega)$ is a morphism. If β is d_ω-closed, then by (7) $\mathrm{e}^{-f_i}\beta$ is closed in U_i, where f_i is a scaling function. In particular $\beta_{|U_i}$ is locally conformally closed.

Conversely let β be a locally conformally closed form such that the local functions $\sigma_i\colon U_i \to \mathbb{R}$, $i \in I$, are unique up to constant. Then the Cech cocycle $\beta_{|U_i} \in H^*_{d\sigma_i}(M)$ glues into a cocycle $\beta \in H^*_\omega(M)$, where $\omega_{|U_i} = d\sigma_i$.

3.1.2. *Covering spaces and automorphic forms.* Let $\omega \in \Omega^1(M)$ be closed. A covering space $\pi\colon \tilde{M} \to (M,\omega)$ is called exact if $\tilde{\omega} := \pi^*\omega$ is exact. This is equivalent to saying that

$$\pi\colon (\tilde{M},0) \to (M,\omega)$$

is a morphism. Therefore according to (7), $\mathrm{e}^{-f}\pi^*\beta$ is closed whenever $\beta \in \Omega^k(M)$ is d_ω-closed, where f is a scaling function. The *smallest exact covering space* of (M,ω) is the one with fundamental group the kernel of the additive character ω.

Consider the multiplicative character

$$\begin{array}{rcl} \chi\colon \Gamma & \longrightarrow & \mathbb{R}^{>0} \\ \gamma & \longmapsto & \chi(\gamma),\ \gamma^*\mathrm{e}^f = \chi(\gamma)\mathrm{e}^f. \end{array} \tag{8}$$

For every $\beta \in \Omega^*(M)$ the group of deck transformations Γ acts on $\mathrm{e}^{-f}\pi^*\beta$ by homotheties $\gamma \to \chi(\gamma)$. We denote the subcomplex of all forms with that property by $\Omega^*(\tilde{M})^\chi$, and we refer to them as automorphic forms (w.r.t. χ). Note that χ is related with the additive character ω in a straightforward manner: the additive character induces additive character on $\tilde{M}$ by pull back or equivalently by using

$$1 \to \pi_1(\tilde{M}) \to \pi_1(M) \to \Gamma \to 1, \tag{9}$$

that we still denote by the same name, and one has

$$-\ln\chi = \omega. \tag{10}$$

In particular this allows to read the additive character by data in the exact covering space (see also [32]).

Conversely, we say that $\tilde{\beta} \in \Omega^*(\tilde{M})$ is *automorphic* if the group of deck transformations acts by homotheties on $\tilde{\beta}$. We denote by $\chi_{\tilde{\beta}}$ the corresponding character. Using (9), the character induces a character in M, and taking minus its logarithm an additive one, that is an element in $H^1(M,\mathbb{Z})$. Let $\omega_{\tilde{\beta}} \in \Omega^1(M)$ be a representative. Then

$$\pi\colon (\tilde{M},0) \to (M,\omega_{\tilde{\beta}})$$

is a morphism. Let f be a scaling function for $\pi^*\omega_{\tilde{\beta}}$. Then by (10) $\mathrm{e}^f\tilde{\beta}$ is invariant under the action of Γ, and thus descends to $\beta \in \Omega^k(M)$ which is $d_{\omega_{\tilde{\beta}}}$-closed if $\tilde{\beta}$ is closed.

We summarize this correspondence in a lemma for its latter use (see also [3]).

Lemma 1. *Let $\pi\colon \tilde{M} \to (M,\omega)$ be a exact covering space. Then it determines a character χ such that the assignment*

$$\begin{array}{rcl} \Omega^*(M,d_\omega) & \longrightarrow & \Omega^*(\tilde{M},d)^\chi \\ \beta & \longmapsto & \mathrm{e}^{-f}\tilde{\beta}, \end{array} \tag{11}$$

where f is a scaling function, is a monomorphism of chain complexes sending forms into automorphic forms. To avoid the choice of scaling function one may speak of a monomorphism from the homothety class of $\Omega^*(M,d_\omega)$ to the homothety class of the subcomplex $\Omega^*(\tilde{M},d)^\chi$, which descends to homothety classes of twisted de Rham cohomology.
Conversely, any character $\chi\colon \Gamma \to \mathbb{R}^{>0}$ determimes a cohomology class of $H^1(M,\mathbb{R})$, and for any representative ω_χ a chain map

$$\Omega^*(\tilde{M},d)^\chi \longrightarrow \Omega^*(M,d_{\omega_\chi}). \tag{12}$$

To get rid of choices one speaks of a well defined map form the homothety class of $\Omega^*(\tilde{M},d)^\chi$ into the conformal class of $\Omega^*(M,d_{\omega_\chi})$. Clearly, both constructions are inverse of each other (when we consider conformal classes of twisted de Rham complexes in M).

3.2. **Computations of twisted de Rham cohomology.** As for computations of twisted de Rham cohomology (for ω non-exact), these are hard. If M is connected $H^0_\omega(M) = 0$, and if additionally M is compact and orientable then $H^{\mathrm{top}}_\omega(M) = 0$ [3, 11, 12]. Under the compactness and orientability assumptions, because the twisted differential is a degree zero deformation of the de Rham differential, the Euler characteristic of the twisted de Rham complex is the Euler characteristic of

M. If one further assumes that ω is parallel for some Riemannian metric (and non-trivial), then the twisted de Rham complex is acyclic [17]. There are some explicit computations by Banyaga describing non-trivial twisted cohomology classes in a particular 4-manifold [1, 4].

Our contribution to computations of twisted de Rham cohomology will be showing that the degree 2 (conformal) twisted de Rham cohomology class associated to the so called Oeljeklaus-Toma l.c.K manifolds is non-trivial. We postpone the proof to the appendix A (proposition 3), once the necessary material on l.c.K. structures has been introduced.

Remark 4. It is natural to extend to the twisted setting geometries defined by conditions on forms and their exterior differentials. Thus, in order to get new examples of such structures one would like to have simple topological constructions to produce new twisted cohomology classes in a fixed degree. Unfortunately, these seem difficult to come up with (for example, given (M, ω) and (M', ω'), it is natural to consider $(M\times M', \omega+\omega')$; it is true that if $d_\omega\beta = d_{\omega'}\beta' = 0$, then $d_{\omega+\omega'}(\beta\wedge\beta') = 0$, but the degree is increased).

4. Locally conformal symplectic structures

Recall that an almost symplectic manifold (M, Φ) is said to be l.c.s. if Φ is locally conformal closed ([11, 34]). If we are in dimension greater than 2, an assumption which we make from now on, isotropic subspaces cannot have codimension 1, so a l.c.s. manifold is given by a closed 1-form ω, the Lee form, and a maximally non-degenerate d_ω-closed 2-form Φ. In other words, a l.c.s. form should be understood as a symplectic form in an appropriate twisted de Rham complex. The cohomology class of the Lee form is the *Lee class* of (M, Φ). The rank and period lattice of the l.c.s. structure are the rank and period lattice of its Lee class. Several of our results are stated for l.c.s. structures with integral period lattice, but they remain valid for discrete lattices.

Non-degeneracy is clearly a conformal property, and thus it is natural to consider conformal classes of l.c.s. structures. In this respect, it is worth pointing out that in l.c.s. geometry one is often able to get results at the level of conformal classes. A good illustration of this fact is the Moser stability result in [1] (see theorem 7), and the reduction by group actions in [14]. Of course, it is much desirable to prove statements at the level of homothety classes or even of l.c.s. forms when possible.

A l.c.s. manifold (M, Φ) is called *exact* if Φ is d_ω-exact, where ω is the Lee form of (M, Φ). We will use the notation (Φ, α) for an exact l.c.s. structure Φ with fixed potential 1-form α. Of course, the information given by either of the tuples $(\Phi, \alpha), (\Phi, \alpha, \omega)$ is the same, so we often omit the Lee form.

Very much as in symplectic geometry a l.c.s. form Φ induces a vector bundle isomorphism $\flat_\Phi : TM \to T^*M$, given by

$$\flat_\Phi(v)(x) = i_v\Phi(x) \text{ for } x \in M \text{ and } v \in T_xM. \tag{13}$$

Now, we consider the Lie algebra of infinitesimal automorphisms of (M, Φ), i.e.,

$$\mathfrak{X}_\Phi(M) = \{X \in \mathfrak{X}(M)/\mathcal{L}_X\Phi = 0\}.$$

Since Φ is non-degenerate, we deduce that for all $X \in \mathfrak{X}_\Phi(M)$, $\mathcal{L}_X\omega = 0$, i.e., $\omega(X) = \textit{constant}$. Moreover, from $d\omega = 0$, we obtain that $\omega([X, Y]) = 0$, for all $X, Y \in \mathfrak{X}_\Phi(M)$. Thus, we have the Lie algebra morphism

$$l : \mathfrak{X}_\Phi(M) \to \mathbb{R}, \quad l(X) = \omega(X)$$

where on $\mathbb{R}$ one takes the commutative Lie algebra structure. In particular the *anti Lee vector field*

$$E := -\flat_\Phi^{-1}(\omega) \tag{14}$$

is in the kernel of l.

If $l \neq 0$ then we say that (M, Φ) is a *l.c.s. manifold of the first kind*. A choice of *transverse infinitesimal automorphism (t.i.a.)* $B \in l^{-1}(1)$ produces a 1-form via the formula

$$\alpha = -\flat_\Phi(B), \tag{15}$$

and it can be checked that

$$d_\omega \alpha = \Phi.$$

Therefore l.c.s. structures of the first kind are in particular exact. We note that being of the first kind -unlike exactness- is a property of the homotethy class of the l.c.s. structure, but not of the conformal class in general [34] (so in particular being exact is weaker than being of the first kind). We also remark that $i_{[B,E]}\Phi = 0$, which implies that $[B, E] = 0$.

Remark 5. Another way of arriving at the l.c.s. structures of the first kind among exact ones is as follows: Consider (M, Φ) an exact l.c.s. structure and select α a potential 1-form. In analogy with symplectic geometry one defines a vector field by $\alpha = -\flat_\Phi(B)$ and expects an special behaviour of Φ under the flow if B. But one gets

$$\mathcal{L}_X \Phi = (1 - \omega(B))\Phi,$$

and obtains either a Liouville type condition or a symplectic type condition by imposing $\omega(B) = 0$ or $\omega(B) = 1$ (actually $\omega(B) \neq 0$, but B is rescalled to give 1). We are mainly interested in compact l.c.s. structures, so the Liouville type condition is impossible since Φ^n is a volume form. Thus, the symplectic type condition, which coincides with being a l.c.s. structure of the first kind, appears as the relevant subclass of exact l.c.s. structures on compact manifolds.

Remark 6. L.c.s. structures of the first kind in M^{2n} are discussed under the name of contact pairs of type $(n-1, 0)$ in [2].

An example of l.c.s. manifold of the first kind is $(S^{2n-1} \times S^1, \Phi_n, \eta_n, d\theta)$, where the canonical integral 1-form on the circle $d\theta$ is the Lee form, and the contact 1-form η_n on S^{2n-1} is the potential 1-form (so $\Phi_n := d_{d\theta}\eta_n$).

The statement of theorem 1 -whose prove we are ready to give- is that $(S^{2n-1} \times S^1, \Phi_n, \eta_n, d\theta)$ are universal manifolds for exact l.c.s. structures with integral period lattice on compact manifolds.

Proof of theorem 1. By hypothesis we can write

$$\Phi = d_\omega \alpha.$$

Using theorem 5, we deduce that for any natural $N \geq 4n + 2$ there exist an embedding $\Psi_1 : M \to S^{2N-1}$ and a real number $c, c > 0$, such that

$$\Psi_1^*(c\eta_N) = \alpha. \tag{16}$$

From the integrality assumption on the Lee form we conclude the existence of a smooth map $\tau : M \to S^1$ such that

$$\tau^*(d\theta) = \omega. \tag{17}$$

Now, the embedding $\Psi : M \to S^{2N-1} \times S^1$ given by

$$\Psi(x) = (\Psi_1(x), \tau(x))$$

satisfies (2), and this proves theorem 1.

Note that $\Psi\colon (M, \omega) \to (S^1 \times S^{2N-1}, d\theta)$ is a strict morphism which must be full since ω has integral period lattice and morphisms are rank decreasing. And by construction the homothety class of η_N is pulled back to α. $\square$

Remark 7. Theorem 1 remains true when the periods of the Lee form generate the discrete lattice $q\mathbb{Z}$, $q \in \mathbb{R}^{>0}$. One needs to use instead the l.c.s. structures of the first kind $(S^{2N-1} \times S^1, \Phi_{N,q}, q\eta_N, d\theta)$, where $\Phi_{N,q} = q\Phi_N$.

Remark 8. If (M, Φ, α) is a l.c.s. structure of the first kind with t.i.a. B related to α as in (15), then the Lee form is no-where vanishing. Therefore it defines a foliation without holonomy, the discreteness of the period lattice being equivalent to the foliation being a fibration over S^1. The restriction of α to each leaf is a contact 1-form. So the l.c.s. structure of the first kind can be understood as a 1-parameter family of exact contact manifolds with a transverse automorphism (the integration of the t.i.a). With this description theorem 1 for l.c.s. structures of the first kind is the appropriate 1-parameter version of theorem 5 for contact forms.

Proof of corollary 1. Under the hypotheses of the corollary, we have that

$$H^k_\omega(M) = \{0\}, \quad \text{for all } k$$

(see theorem 4.5 in [17]). Thus, the l.c.s. structure of M is exact and we may apply theorem 1. $\square$

Remark 9. It is natural to define universal models for l.c.s. on compact manifold by requiring the existence of embeddings into them which are (1) full (strict) morphisms, and (2) pull back the (strict) conformal class of the l.c.s. form into the given one. This would imply the existence of a moduli parametrized by lattices $\Lambda \subset \mathbb{R}$. As a consequence one would need to have a large supply of compact l.c.s. manifolds with arbitrary period lattice, but examples are scarce. In this respect it is noteworthy the family of Oeljeklaus-Toma l.c.K. structures which have arbitrary rank [32], and are non-exact as will be shown in appendix A (note that our universal models for reduction are exact l.c.s. manifolds with arbitrary period lattices, but they are non-compact). As an illustration of the difficulty of producing examples of l.c.s. structures consider a contact manifold (N, η) and the associated exact l.c.s. manifold $(S^1 \times N, d_{d\theta}\eta, d\theta)$. Now let Σ be an orientable surface. For the product manifold $S^1 \times N \times \Sigma$ finding a 1-form α such that $d_{d\theta}\alpha$ is l.c.s. implies finding contact structures in $\Sigma \times N$. This is a very non-trivial problem whose solution was only found recently [5]. It is natural then to ask wheter given a l.c.s. manifold (M, Φ), one can endow $\Sigma \times M$ with a l.c.s. structure (in particular one would be overcoming the problems noted in remark 4 about producing new closed 2-forms in twisted de Rham complexes out of old ones).

5. Relation with embedding results for l.c.K. manifolds

In this section we will discuss the relation between theorem 1 and recent embedding results by Ornea and Verbitsky for locally conformal Kähler (l.c.K.) structures.

5.1. Vaisman manifolds.

A l.c.K. structure on a complex manifold (M, J) is given by a Hermitian metric g which is locally conformal to a Kähler one. The underlying l.c.s. structure is defined by the associated 2-form

$$\Phi_g := g(\cdot, J\cdot).$$

Equivalently, a l.c.K. structure is given by a l.c.s. form Φ and an integrable compatible almost complex structure J. If $(\tilde{M}, \tilde{J}) \to (M, J)$ is a complex covering space, according to lemma 1 there is a one to one correspondence between homothety classes of automorphic Kähler forms Ω on $(\tilde{M}, \tilde{J})$, and conformal classes of l.c.K. structures in (M, J) whose Lee class becomes exact in $\tilde{M}$.

Let (M, J, Φ_g) be a l.c.K. manifold with Lee form ω. The *Lee vector field* B is the metric dual of the Lee form. By construction $-JB$ is the anti-Lee vector field of the underlying l.c.s. as defined in (14). A l.c.K. structure is called *Vaisman* if the Lee

form is parallel. It follows that $B, E = -JB$ are both Killing, preserve J, and have commuting flows ($B - iJE$ is a holomorphic vector field) [15]. If we go to an exact complex covering space $(\tilde{M}, \tilde{J})$, and we let Ω be in the homotethy class of Kähler forms furnished by lemma 1, the lift of the flow of the Lee vector field is by Kähler homotheties. Unlike the case of l.c.K. structures the data $(\tilde{M}, \tilde{J}, \Omega)$ determines uniquely the homotethy class of the Vaisman structure (M, J, Φ_g). Thus, one can take this second approach as a definition of Vaisman structure.

By definition the Lee vector field of a Vaisman structure belongs to $\mathfrak{X}_{\Phi_g}(M)$. Therefore Vaisman structures are the natural analogs in l.c.K. geometry of l.c.s. structures of the first kind (see also [2], where compatible almost complex structures are also brought into the picture). It should be noted though, that Vaisman structures on a compact manifold have always rank 1 [25]. For simplicity we will normalize our Vaisman structures so that $l(B) = 1$, i.e. we can take the Lee vector field as t.i.a., and thus $-\flat_{\Phi_g}^{-1}(B)$ is a potential 1-form for the l.c.s. structure Φ_g (and in this way we fix a representative of the homothety class).

Examples of Vaisman manifolds are the diagonal Hopf manifolds (H_A, J_A, Φ_A) [15]. One rather introduces these Vaisman structures by starting with the covering space $(\tilde{H}_A, \tilde{J}_A) := (\mathbb{C}^N \backslash \{0\}, J_{\mathrm{std}})$, and taking $\Gamma \cong \mathbb{Z}$ generated by the linear action of an invertible matrix A which has all its eigenvalues with norm < 1, and which is diagonalizable. Because of the conditions on its eigenvalues A is in the image of the exponential map and has a unique logarithm. That defines a 1-parameter group of holomorphic transformations whose time 1 map is the action by A. In [15], section 3, a family $\Omega_{A,q}$, $q \in \mathbb{R}^{>0}$, of Kähler forms for which the previous flow acts by Kähler homotheties is given. The unique normalized Vaisman structure in (H_A, J_A) is denoted by $\Phi_{A,q}$. The parameter q is such that the period lattice of the Lee form is $q\mathbb{Z}$. Diagonal Hopf manifolds are diffeomorphic to $S^{2N-1} \times S^1$ (see the discussion in the proof of proposition 1). The standard l.c.s. structure in theorem 1 is the one associated to a diagonal Hopf manifold where $A = \lambda \mathrm{Id}$ is a real multiple of the identity (for an obvious diffeomorphism between $H_{\lambda \mathrm{Id}}$ and $S^{2N-1} \times S^1$).

5.2. **L.c.K structures with automorphic potential.** Theorem 1 holds not just for l.c.s. structures of the first kind with integral period lattice, but for exact ones. In [26] (see also [27]) Ornea and Verbitsky have introduced the notion of l.c.K. structure with vanishing Bott-Chern class (or with automorphic potential): given any closed 1-form ω, in the presence of a complex structure the complexified twisted de Rham complex $(\Omega^*(M, \mathbb{C}), d_\omega)$ can be split into its holomorphic and antiholomorphic components, and so the twisted differential

$$d_\omega = \partial_\omega + \bar{\partial}_\omega.$$

This gives rise to a Bott-Chern cochain complex with cohomology groups $H^{p,q}_{\partial_\omega \bar{\partial}_\omega}(M)$, (see [26] for details); the action of functions in (6) on twisted de Rham complexes induces an action on Bott-Chern cochain complexes. Suffice it to say here that for a l.c.K manifold (M, J, Φ_g) the l.c.K. 2-form defines the *Bott-Chern class*

$$[\Phi_g] \in H^{1,1}_{\partial_\omega \bar{\partial}_\omega}(M),$$

and that the identity induces a homomorphism

$$H^{1,1}_{\partial_\omega \bar{\partial}_\omega}(M) \to H^2_\omega(M)$$

sending the Bott-Chern class to the class $[\Phi_g] \in H^2_\omega(M)$ (one has the usual equality $i\partial_\omega \bar{\partial}_\omega = d_\omega d^c_\omega$, with $d^c_\omega = J^* d_\omega$).

A l.c.K. structure is said to have *vanishing Bott-Chern class* if $[\Phi_g] \in H^{1,1}_{\partial_\omega \bar{\partial}_\omega}(M)$ is trivial. Having vanishing Bott-Chern class is conformally invariant, and it also implies that the class $[\Phi_g] \in H^2_\omega(M)$ is trivial, so the underlying l.c.s. structure is

exact. It is not clear whether the converse is true or not, this being related to the existence of a global $\partial_\omega\bar{\partial}_\omega$ lemma [26].

A Vaisman structure has vanishing Bott-Chern class, and a canonical $d_\omega d^c_\omega$-potential is given by the constant function 1. Indeed, the equation

$$\partial_\omega\bar{\partial}_\omega 1 = \Phi_g$$

is equivalent to

$$d(J^*\omega) = \Phi_g + \omega \wedge J^*\omega,$$

which in turn is equivalent to

$$\mathcal{L}_B\Phi_g = i_B d\Phi_g + d i_B\Phi_g = 0.$$

For a exact complex covering space $(\tilde{M},\tilde{J})$ of (M,J,ω), in the assignment described in lemma 1 $d_\omega d^c_\omega$-potentials for Φ_g correspond to automorphic usual dd^c-potentials for the automorphic Kähler form associated to the choice of scaling function. If M is compact and $\tilde{M}$ is the smallest exact covering space then the dd^c-potential is proper [29].

One advantage of l.c.K. structures with automorphic potential is that one may construct new ones via small perturbations. For example going to a exact covering space $\tilde{M}$, one can perturb a bit the complex structure in the base, lift it, and perturb the initial potential a little bit so that the Kähler condition still holds (and one can also allow for perturbations of the subgroup Γ so that the Lee class can change).

Examples of l.c.K. structures with automorphic potential are constructed in the linear Hopf manifolds (H_A, J_A). This is the same construction as for diagonal Hopf structures, but the invertible matrix A is just supposed to have eigenvalues of norm < 1. The automorphic potential is constructed by perturbation as indicated for example in [28] (see also [8]): the closure of the orbit of any A as above (by the action by conjugation of the complex general linear group) contains diagonalizable matrices A'. This implies the existence of a diffeomorphism $H_A \cong H_{A'}$, pushing J_A in to a complex structure in $H_{A'}$ that we still denote J_A. One can assume that J_A and $J_{A'}$ are as close as desired. Thus the same automorphic potential in the covering space $(\tilde{H}_{A'}\tilde{J}_{A'})$ for the Vaisman structure $\Phi_{A'}$ defines an automorphic Kähler metric for the lift of the complex structure J_A. This gives rise to a l.c.K. form with automorphic potential Φ_A in $(H_{A'}, J_A)$ (and hence in (H_A, J_A)) by using the fixed Lee form in $H_{A'}$ and the fixed scaling function in $\tilde{H}_{A'}$.

Remark 10. Observe that for the given diffeomorphism the Lee form is the same for both Φ_A and $\Phi_{A'}$; also the automorphic potential in the covering space is chosen to be the same. Note as well that one can arrange for the existence of A_t, $t \in [0,1]$, $A_0 = A'$, $A_1 = A$ and so that the construction holds with parameters (i.e. on has $(H_{A'}, J_{A_t}, \Phi_{A_t})$, $t \in [0,1]$ l.c.K. structures with the same automorphic potential (in the covering space) and the same Lee form).

The main embedding result of Ornea and Verbitsky is the following:

Theorem 6. [26] *Let (M,J,Φ_g,r) be a l.c.K. structure with automorphic potential on a compact manifold of complex dimension at least 3. Then there exists a holomorphic embedding of (M,J) into a linear Hopf manifold (H_A,J_A). Moreover, if (M,J,Φ_g) is Vaisman then there exist a holomorphic embedding into a diagonal Hopf manifold.*

Because we want to eventually prove theorem 2 we will work with l.c.K. structures with integral period lattice.

5.3. **Families of l.c.K. structures and Moser type results.** Extending previous work of Banyaga [3], Bande and Kotschick [1] have proved a Moser stability type result for l.c.s. structures. Here we only state a particular case which will suffice for our purposes.

Theorem 7. *(Corollary 3.3. in* [1]*). Let Φ_t, $t \in [0,1]$, be a smooth family of l.c.s. structures on a compact manifold M such that the corresponding Lee forms ω_t have the same de Rham cohomology class. Suppose there exists a smooth family of 1-forms α_t such that $\Phi_t = d_{\omega_t}\alpha_t$. Then there exists an isotopy ϕ_t such that $\phi_t^*\Phi_t$ is conformally equivalent to Φ_0 for all t.*

In a compact Kähler manifold the convex combination of any two Kähler forms is Kähler, so cohomologous Kähler forms define the same symplectic structure up to a diffeomorphism isotopic to the identity.

The space of l.c.K. structures with fixed Lee form is discussed in [26]. One can slightly generalize those results to obtain stability for the conformal class of the underlying l.c.s. structures.

Lemma 2. *Let (M,J) be a complex manifold. Then the space of l.c.K. structures with fixed Lee class is connected. Moreover, the same holds for l.c.K. structures with automorphic potential. In particular if M is compact the action of the group of diffeomorphisms isotopic to the identity on the conformal classes of l.c.s. structures with l.c.K. representatives with automorphic potential, has orbits parametrized by the Lee class.*

Proof. Let $\Phi_g, \Phi_{g'}$ be two l.c.K. structures in (M,J) with the same Lee class $[\omega]$. Let $(\tilde{M},\tilde{J})$ be a exact covering space. Then one has scaling functions $f, f' \in C^\infty(\tilde{M})$ and automorphic Kähler forms $\Omega = e^{-f}\tilde{\Phi}_g, \Omega' = e^{-f'}\tilde{\Phi}_{g'}$.

The convex combination $\Omega_t = (1-t)\Omega + t\Omega'$ defines a family of automorphic Kähler forms (for $\chi(e^{-[\omega]})$). Thus, they define a 1-parameter family of conformal classes of l.c.K. structures with Lee class $[\omega]$. It is easy to find a smooth path of representatives by just choosing the path of functions $f_t = (1-t)f + tf'$ which have all additive character $[\omega]$.

If the given l.c.K. structures have automorphic potential r, r', then $r_t = (1-t)r + tr'$ is an automorphic potential for Φ_{g_t}.

Thus, if we are in a compact manifold we can apply theorem 7 and this proves the lemma. □

Remark 11. The stability result in lemma 2 also holds for exact l.c.K. structures by applying Hodge theory to find potential 1-forms [1].

All linear Hopf manifolds H_A are diffeomorphic to $S^{2N-1}\times S^1$. We want to show that, in an appropriate sense, the conformal class of the l.c.s. structure induced by any Φ_A with integral period lattice is unique (see remark 12).

Proposition 1. *If $A \in GL(N,\mathbb{C})$ has eigenvalues of norm smaller than 1, and $(H_A, J_A, \Phi_A, \omega_A)$ is a l.c.K. structure with automorphic potential and integral period lattice as constructed in* [15, 28]*, then there exists a diffeomorphism*

$$\phi_A \colon (S^{2N-1}\times S^1, d\theta) \to (H_A, \omega_A)$$

which is a (full) morphism and pulls back the conformal class of Φ_A to the conformal class of the standard l.c.s. structure with integral period lattice Φ_N.

Proof. According to theorem 7 we just need to find a diffeomorphism such that $\phi_A^*\Phi_A$ and Φ_N can be joined by a (piecewise) smooth path of l.c.K. structures -for possibly different complex structures- with integral period lattice, and with potential 1-forms varying smoothly.

We do it in several steps. We assume that A is not diagonalizable. By remark 10 we can find A' diagonalizable and a diffeomorphism $\phi_1\colon H_{A'} \to H_A$ so that $\Phi_{A'}$ and $\phi_1^*\Phi_A$ can be joined by a smooth family of l.c.K. structures with automorphic potential and integral period lattice, and such that the Lee form and automorphic potential are the same. Therefore, we can assume without loss of generality that A is diagonalizable. Note as well that the deformation argument in [28] may produce many l.c.K. structures with automorphic potential and integral period lattice. By lemma 2 they all belong to the same conformal class.

The second step amounts to comparing all l.c.K. Vaisman structures with integral period lattice in a given $GL(N,\mathbb{C})$ orbit. In [15] for A diagonal an automorphic potential is given defining a Vaisman structure with integral period lattice. The associated Kähler metric in $\mathbb{C}^N\backslash\{0\}$ is invariant by conjugation by $GL(N,\mathbb{C})$, since it is defined by a potential. Given any A, A' in the same orbit, if $A' = L * A$, then L pushes any path of l.c.K. structures with automorphic potential and integral period lattice starting at the Vaisman structure Φ_A, into a path of l.c.K. structures with automorphic potential and integral period lattice starting at the Vaisman structure $\Phi_{A'}$. This implies that we can assume A to be diagonal.

The automorphic potentials for the Vaisman structures with integral period lattice in [15] in diagonal Hopf manifolds, depend smoothly on the eigenvalues (see also [26], section 2.2, for an explicit formula). Let A be diagonal. We assume that all eigenvalues have norm q . We let $A' = A/2q$ and consider the convex combination $A_t = (1-t)A + tA'$ with eigenvalues whose norm is q_t.

A fundamental domain for H_{A_t} is the closed annulus $\mathbb{A}(q_t, 1)$ of Euclidean radii $q_t, 1$, and the manifold is obtained by applying the same diffeomorphism $S^{2N-1} \to S^{2N-1}$ for all t. Let $k_t\colon [q_t, 1] \to [1/2, 1]$ be the linear orientation preserving diffeomorphism. Then for the product decomposition of the annuli into radial and spherical coordinates,

$$k_t \times \mathrm{Id}\colon \mathbb{A}(g_t, 1) \to \mathbb{A}(1/2, 1),$$

is a diffeomorphism which descends to a diffeomorphism

$$K_t\colon H_{A_t} \to H_{A'}.$$

Therefore $K_{t*}\Phi_{A_t}$ is a path of l.c.K. structures (for the induced complex structures) with automorphic potential and integral period lattice which connects $\Phi_{A'}$ with $K_{0*}\Phi_A$. Then by Moser stability the positive conformal class is the same, which implies that we may scale the eigenvalues at will. If not all eigenvalues have the same norm, they do have the same norm for an obvious Hermitian metric. By connecting that metric with the Euclidean one, and using the corresponding path of integral Vaisman structures, we may assume without loss of generality that all eigenvalues have norm $1/2$.

We note that in the previous considerations we really need the path of l.c.K. structures connecting the two sets of eigenvalues; the diffeomorphisms that we are considering between different diagonal Hopf manifolds are not holomorphic in general, so one cannot apply lemma 2.

The final step is correcting the argument of the eigenvalues. If we identify the spheres of radius 1, 1/2 by the homothety, we obtain the isomorphism

$$(H_{\mathrm{Id}/2}, \Phi_{\mathrm{Id}/2}, d^c_{\omega_{\mathrm{Id}/2}}, \omega_{\mathrm{Id}/2}) \cong (S^{2N-1} \times S^1, \Phi_N, \eta_N, d\theta).$$

Also H_A is a mapping torus over S^1 with return map $\varphi_A\colon S^{2N-1} \to S^{2N-1}$, which is clearly isotopic to the identity. We can for example construct a path of diagonal matrices A_t joining A with $\mathrm{Id}/2$ by rotating the eigenvalues clockwise until we reach $1/2$. It is easy to produce diffeomorphisms

$$K_t\colon H_{A_t} \to H_{\mathrm{Id}/2}$$

thus getting a path of l.c.K. structures (for the induced complex structure) with automorphic potential and integral period lattice connecting Φ_N with $K_{0*}\Phi_A$. □

Remark 12. Linear Hopf manifolds are not canonically diffeomorphic to $S^{2N-1} \times S^1$. Proposition 1 has produced for H_A many diffeomorphisms $H_A \to H_{\mathrm{Id}/2}$ in the same isotopy class taking Φ_A to the positive conformal class of Φ_N. Therefore a fortiori the diffeomorphism class of the positive conformal classes of Φ_A and Φ_N coincide, and it is in this sense that the positive conformal class of Φ_A in $S^{2N-1} \times S^1$ is unique.

Proof of theorem 2. Let (M, J, Φ_g) be the given l.c.K. structure with automorphic potential and integral period lattice. Suppose that Ψ is a holomorphic embedding of (M, J) into the linear Hopf manifold (H_A, J_A) with a l.c.K. structure Φ_A with automorphic potential r_A, and Lee form ω_A with integral period lattice (see [27]).

By proposition 1 we have $\phi\colon H_A \to S^{2N-1} \times S^1$ a diffeomorphism pulling back Φ_N to the positive conformal class of Φ_A. Because $\Psi^*\colon (M, \Psi^*\omega_A) \to (H_A, \omega_A)$ is a morphism, $\Psi^*\Phi_A$ is $d_{\Psi^*\omega_A}$-closed. Non-degeneracy follows from the fact that $\Psi^*\Phi_A(\cdot, J\cdot)$ is the restriction of the Hermitian metric $\Phi_A(\cdot, J_A\cdot)$. Therefore, by lemma 2 we just need to show that $[\Psi^*\omega_A] = [\omega_g]$ and that $\Psi^*\Phi_A$ is a l.c.K. structure with automorphic potential.

To this end we need to recall some aspects of the construction of the holomorphic embedding: The manifold (M, J) carries a l.c.K. structure with integral period lattice. In the smallest exact covering space $(\tilde{M}, \tilde{J}, \Omega)$, whick has deck transformation group $\Gamma \cong \mathbb{Z}$ generated by the contraction γ, one constructs a holomorphic map from the 1-point Stein compactifications

$$\tilde{\Psi}\colon \hat{\tilde{M}} \to \mathbb{C}^n$$

which is equivariant with respect to the group isomorphism $\Gamma \to \langle A \rangle$, $\gamma \mapsto A$. Then one gets the (holomorphic) commutative diagram of morphisms

$$\begin{array}{ccc} (\tilde{M}, 0) & \xrightarrow{\tilde{\Psi}} & (\mathbb{C}^N \backslash \{0\}, 0) \\ {\scriptstyle \pi_\gamma}\big\downarrow & & {\scriptstyle \pi_A}\big\downarrow \\ (M, \Psi^*\omega_A) & \xrightarrow{\Psi} & (H_A, \omega_A) \end{array} \tag{18}$$

By lemma 1 the additive character $[\omega_A]$ characterizes the multiplicative character χ_A. Because $\tilde{\Psi}$ is equivariant with respect to the action of the groups of deck transformations and the morphism relating the deck transformation groups is an isomorphism, we have $\Psi^*\chi_A = \chi_{[\omega_g]}$. By lemma 1 and commutativity of (18) we conclude

$$[\Psi^*\omega_A] = [\omega_g].$$

Because $\tilde{\Psi}$ is equivariant w.r.t. deck transformations, it pulls back an automorphic potential for Ω_A (w.r.t. $\chi_{[\omega_A]}$) into an automorphic potential for $\tilde{\Psi}^*\Omega_A$ (w.r.t. $\chi_{[\Psi^*\omega_A]}$). Therefore by commutativity of (18) it follows that the conformal class of l.c.K. structures defined by $\Psi^*\Phi_A$ has automorphic potential.

□

Remark 13. Theorem 2 holds more generally for arbitrary l.c.K. structures with automorphic potential (see remark 7).

6. Universal models for reduction of l.c.s. structures of the first kind

Any symplectic structure in a manifold of finite type can be obtained by reduction of the standard symplectic structure in $\mathbb{R}^{2n}$ [10].

Reduction has been extended for l.c.s. structures [14]. Among l.c.s. structures, those of the first kind bear relations with contact and cosymplectic geometry (see remark 8). Based on the universal models for reduction for the latter structures, one is led to consider the following family of l.c.s. manifolds of the first kind: For each pair of natural numbers (N, k), we define

$$M_{k,N} = \mathbb{R} \times \mathcal{J}^1(\mathbb{T}^k \times \mathbb{R}^N), \tag{19}$$

where $\mathcal{J}^1(\mathbb{T}^k \times \mathbb{R}^N)$ is the 1-jet bundle of the cartesian product of the k-dimensional torus and $\mathbb{R}^N$. Denote by s the coordinate on the first factor in (19), by u the real coordinate in $\mathcal{J}^1(\mathbb{T}^k \times \mathbb{R}^N) = \mathbb{R} \times T^*(\mathbb{T}^k \times \mathbb{R}^N)$, by $(t_1, \ldots, t_N)$ the coordinates in $\mathbb{R}^N$, and by $\theta_1, \ldots, \theta_k$ the periodic coordinates in $\mathbb{T}^k$ (with period 1). If $\mu = (\mu_1, \ldots, \mu_k)$ is a k-tuple of real numbers, a computation shows that the 1-forms

$$\omega_\mu = ds + \sum_{j=1}^{k} \mu_j d\theta_j, \; \alpha_{k,N} = du - \lambda_{\mathbb{T}^k \times \mathbb{R}^N}$$

fit into a l.c.s. structure of the first kind

$$\Phi_{k,N,\mu} := d\alpha_{k,N} - \omega_\mu \wedge \alpha_{k,N}.$$

The t.i.a. associated to $\alpha_{k,N}$ and the anti-Lee vector field are respectively

$$B = \frac{\partial}{\partial s}, \quad E = -\frac{\partial}{\partial u}.$$

Let (M, Φ, α) and (M', Φ', α') be l.c.s. manifolds of the first kind. A diffeomorphism Ψ is said to be of the first kind if it is a strict morphism and satisfies $\Psi^*\alpha' = \alpha$. In such a case, we have that $\Psi^*\Phi' = \Phi$, and the associated t.i.a. and the anti-Lee vector fields are Ψ-related.

The action of $SL(k, \mathbb{Z})$ in k-tuples of real numbers μ is seen to induce an action on $(M_{k,N}, \Phi_{k,N,\mu}, \alpha_{k,N})$ by diffeomorphisms of the first kind.

Before proving that $(M_{k,N}, \Phi_{k,N,\mu}, \alpha_{k,N})$ are the universal models we are looking for, we need to say a few words about reduction of l.c.s. structures.

Among the results in [14], conditions mimicking coisotropic symplectic reduction are imposed on a submanifold C of a l.c.s manifold (M, Φ, ω), such that the leaf space associated to the involutive distribution $\ker\Phi_{|C}$ inherits a l.c.s. structure ($(C, \ker\Phi_{|C})$ is a *reductive structure* [14]): let $\mathcal{F}$ be the distribution integrating $\ker\Phi_{|C}$. Assuming $C/\mathcal{F}$ to be a manifold, one wants the projection $(C, \omega_{|C}) \to C/\mathcal{F}$ to become a strict morphism, so $\omega_{|C}$ is asked to be $\mathcal{F}$-basic. Then exactly the same proof used for symplectic coisotropic reduction produces a l.c.s. form in the quotient whose pullback is $\Phi_{|C}$. We are interested in finding further conditions so that l.c.s. structures of the first kind are preserved under reduction.

Lemma 3. *Let (Φ, α) be a l.c.s. structure of the first kind on M with Lee form ω and associated t.i.a. B. Let C be a submanifold of M such that the following properties hold:*

1. *B and E are tangent to C.*
2. *The involutive distribution $\ker\omega_{|C} \cap \ker\alpha_{|C} \cap \ker d\alpha_{|C}$ has constant rank, thus defining a foliation $\mathcal{F}$.*
3. *The leaf space $M_0 = C/\mathcal{F}$ has a manifold structure induced by the projection $\pi\colon C \to M_0$.*

Then M_0 inherits a l.c.s. structure of the first kind (Φ_0, α_0) with Lee form ω_0 characterized by $\pi\colon (C, \omega_{|C}) \to (M_0, \omega_0)$ being a strict morphism such that $\alpha_{|C} = \pi^*\alpha_0$ (and thus $\Phi_{|C} = \pi^*\Phi_0$). The associated t.i.a. and anti-Lee vector fields are the projection of B and E respectively (which are $\mathcal{F}$-projectable).

We say the C is strongly reducible and that (M_0, Φ_0, α_0) is the reduction of (M, α, ω) (by C).

Proof. It is routine to check that $\omega_{|C}, \alpha_{|C}, d\alpha_{|C}$ are $\mathcal{F}$-basic forms, $B_{|C}, E_{|C}$ $\mathcal{F}$-projectable vector fields and that they induce in M_0 a l.c.s. with the stated properties. □

We note that a concatenation of reductions is in an obvious way a reduction in just one stage.

Lemma 4. *If a l.c.s. manifold of the first kind (M_3, Φ_3, α_3) is the reduction of a l.c.s. manifold of the first kind (M_2, Φ_2, α_2) by the submanifold $C_2 \subseteq M_2$ and if (M_2, Φ_2, α_2) is the reduction of a l.c.s. manifold of the first kind (M_1, Φ_1, α_1) by the submanifold $C_1 \subseteq M_1$, then (M_3, Φ_3, α_3) is the reduction of (M_1, Φ_1, α_1) by the submanifold $C_1' = \pi_1^{-1}(C_2)$, where $\pi_1 \colon C_1 \to M_2$ denotes the canonical projection.*

Proof of theorem 3. Let (M, Φ, α) be a l.c.s. structure of the first kind in a manifold of finite type. Let k be its rank and let $\mu \in \mathbb{R}^k$ be a basis of its period lattice Λ.

In a first step we construct a l.c.s. manifold of the first kind with the same period lattice and whose Lee form has an appearance close to ω_μ, together with a strongly reductive submanifold whose reduction is (M, Φ, α).

The finiteness of the first Betti number together with the choice of a basis of the period lattice implies that we can write

$$\omega = \omega_0 + \sum_{j=1}^{k} \mu_j \omega_j,$$

where $[\omega_j]$, $j = 1, \ldots, k$, is integral, the classes $\mu_1[\omega_1], \ldots, \mu_k[\omega_k]$ linearly independent over the integers, and ω_0 is exact. We fix $\tau_j \colon M \to S^1$ such that $\tau_j^* d\theta_j = \omega_j$ and define

$$M_1 = M \times T^*(\mathbb{T}^k).$$

Let (θ_j, r_j) be the corresponding coordinates on $\mathbb{T}^k \times \mathbb{R}^k \cong T^*(\mathbb{T}^k)$. Then it can be checked that the 1-forms

$$\alpha_1 = \alpha + \sum_{j=1}^{k} \mu_j r_j (d\theta_j - \omega_j), \quad \omega_1 = \omega_0 + \sum_{j=1}^{k} \mu_j d\theta_j \tag{20}$$

define a l.c.s. structure of the first kind Φ_1 with t.i.a. and anti-Lee vector field respectively

$$B_1 = B + \sum_{j=1}^{k} (i_B \omega_j) \frac{\partial}{\partial \theta_j}, \quad E_1 = E + \sum_{j=1}^{k} (i_E \omega_j) \frac{\partial}{\partial \theta_j}.$$

We define C_1 to be the image of the embedding

$$\begin{array}{rccc} F \colon & M \times \mathbb{R}^k & \longrightarrow & M_1 \\ & (x, r_j) & \longmapsto & (x, \tau_j(x), r_j). \end{array} \tag{21}$$

A direct computation shows that C_1 is a strong reductive submanifold of (M_1, Φ_1, α_1) and that the reduction is isomorphic to (M, α, ω, B).

In the second step we construct a l.c.s. manifold of the first kind with the same Lee form and whose potential 1-form has an appearance close to $\alpha_{N,k}$, together with a strongly reductive submanifold whose reduction is (M_1, Φ_1, α_1).

We define

$$M_2 = \mathbb{R} \times \mathcal{J}^1 M_1,$$

with coordinate s for the first factor and u for the real factor of the 1-jet bundle. The 1-forms

$$\omega_2 = ds + \omega_1, \quad \alpha_2 = du - \lambda_{M_1} \tag{22}$$

are seen to define a l.c.s. structure Φ_2 with t.i.a. and anti-Lee vector field respectively

$$B_2 = \frac{\partial}{\partial s}, \quad E_2 = -\frac{\partial}{\partial u}.$$

As for any l.c.s. structure of the first kind, the vector fields B_1, E_1 have commuting flows. Denote by C_2 the open subset of $\mathbb{R}^2 \times M_1$ in which the composition of both flows is defined, and embed it in M_2 via the map

$$\begin{array}{rcl} G\colon C_2 & \longrightarrow & M_2 \\ (s,u,x) & \longmapsto & (s,-u,-\alpha_1(x)). \end{array}$$

A direct computation shows that C_2 is a strongly reductive submanifold of (M_2, Φ_2, α_2) whose reduction is isomorphic to (M_1, Φ_1, α_1). In fact, if ϕ and ψ are the flows of the vector fields B_1 and E_1, respectively, then the map

$$C_2 \to M_1, \quad (s,u,x_1) \to \psi_u(\phi_s(x_1))$$

induces an isomorphism between the l.c.s. manifold (M_1, Φ_1, α_1) and the l.c.s. reduction of (M_2, Φ_2, α_2) by the submanifold C_2.

In the third step we seek to simplify the formula for the Lee form. To that end we will not perform any reduction, just apply an appropriate diffeomorphism of M_2.

We take $f_0\colon M \to \mathbb{R}$ with $\omega_0 = df_0$. The Lee form can be written

$$\omega_2 = ds + df_0 + \sum_{j=1}^{k} \mu_j d\theta_j.$$

The diffeomorphism is

$$\begin{array}{rcl} H\colon M_3 := M_2 = \mathbb{R}\times\mathbb{R}\times T^*M_1 & \longrightarrow & M_2 \\ (s,u,\xi,x) & \longmapsto & (s - f_0(x), u, \xi, x). \end{array}$$

If we pullback the l.c.s. structure we obtain

$$\alpha_3 = \alpha_2, \quad \omega_3 = ds + \sum_{j=1}^{k} \mu_j d\theta_j, \quad B_3 = B_2, \quad E_3 = E_2.$$

The final step is a further reduction to make the last simplification of the potential 1-form. We take an embedding $M \hookrightarrow \mathbb{R}^{4n}$ which allows us to consider a new embedding $i'\colon M_1 = M \times T^*\mathbb{T}^k \cong M \times \mathbb{T}^k \times \mathbb{R}^k \hookrightarrow \mathbb{T}^k \times \mathbb{R}^N$, with $N = 4n + k$.

Now, we take the universal l.c.s. manifold of the first kind

$$(M_{k,N}, \Phi_{k,N,\mu}, \alpha_{k,N}).$$

Denote by $\pi\colon M_{k,N} \to \mathbb{T}^k \times \mathbb{R}^N$ be the bundle map projection. We define

$$C_4 = \pi^{-1}(i'(M_1)). \tag{23}$$

A final check shows that C_4 is a strongly reductive submanifold of $(M_{k,N}, \Phi_{k,N,\mu}, \alpha_{k,N})$ whose reduction is isomorphic to (M_3, Φ_3, α_3). Thus, using lemma 4, we prove the theorem for the chosen basis μ.

Different choices of basis are related by the action of $SL(k,\mathbb{Z})$, which acts on the corresponding universal manifolds by diffeomorphisms of the first kind. Thus, we rather use Λ in the notation for our universal manifolds -as in theorem 3- since it is the $SL(k,\mathbb{Z})$-orbit what we look at. $\square$

7. Universal models for equivariant reduction of l.c.s. structures of the first kind

In this last section we will prove theorem 4, an equivariant version of theorem 3. Let (M, Φ, α) be a l.c.s. manifold of the first kind with Lee 1-form ω and $\psi : G \times M \to M$ be an action of a Lie group G on M. The action is said to be a *l.c.s. action of the first kind* if each automorphism ψ_g, $g \in G$, is of the first kind. In such a case, we have that the associated t.i.a. B and the anti-Lee vector field E are G-invariant with respect to ψ.

Now, we consider C a strong reducible submanifold of (M, Φ, α) which is G-invariant with respect to ψ. Denote by $(M_0 = C/\mathcal{F}, \Phi_0, \alpha_0)$ the reduction of (M, Φ, α) by C (see lemma 3). Then, one may easily prove the following result.

Proposition 2. *Let $\psi\colon G \times M \to M$ be a l.c.s. action of the first kind and C be a G-invariant strong reducible submanifold of M. Then, there exists an induced l.c.s. action of the first kind $\psi_0\colon G \times M_0 \to M_0$ of G on the l.c.s. reduced manifold $(M_0 = C/\mathcal{F}, \Phi_0, \alpha_0)$ of the first kind.*

If the conditions of proposition 2 hold $(M_0, \Phi_0, \alpha_0, \psi_0)$ is said to be *the equivariant reduction of (M, Φ, α, ψ) by the submanifold C.*

Next, we will prove theorem 4. For this purpose, we will use the following lemma (see [18]).

Lemma 5. *Let G be a compact and connected Lie group and $\psi : G \times M \to M$ be an action of G on a connected manifold M. Then,*

(1) *If β is a k-form on M with integral cohomology class, the average $\bar{\beta} = \int_G (\psi_g^* \beta) dg$ represents the same integral class, provided dg the invariant Haar measure of total volume 1.*
(2) *If $f : M \to S^1$ is a smooth map and the 1-form $\beta = f^* d\theta$ is G-invariant, then there exists a representation $\varphi\colon G \to S^1$ of G on S^1 such that f is equivariant with respect to usual action of S^1 on itself, that is,*

$$f(\psi_g(x)) = \varphi(g) \cdot f(x), \quad \forall g \in G \text{ and } x \in M.$$

Proof of theorem 4. In order to prove this theorem, we will rewrite the proof of theorem 3, adding the corresponding equivariant notions. So, like in the proof of theorem 3, we start with a decomposition of Lee 1-form ω associated with (Φ, α)

$$\omega = \omega_0 + \sum_{j=1}^{k} \mu_j \omega_j.$$

From lemma 5, one deduces that ω_j and the average $\bar{\omega}_j = \int_G \psi_g^*(\omega_j) dg$ represent the same integral cohomology class, for $j = 1, \dots, k$. Therefore, one may suppose without loss of generality, that ω_j is G-invariant.

Now, we may consider the map $\tau_j\colon M \to S^1$ which satisfies $\tau_j^* d\theta_j = \omega_j$, for $j = 1, \dots, k$. Then, using again lemma 5, we can choose for each j an action of G on S^1 (induced by a representation $\varphi_j\colon G \to S^1$) such that the map τ_j is equivariant, i.e.,

$$\tau_j(\psi_g(x)) = \varphi_j(g) \cdot \tau_j(x).$$

We remark that $d\theta_j$ is G-invariant with respect to the action $\bar{\psi}_k\colon G \times \mathbb{T}^k \to \mathbb{T}^k$ given by

$$(\bar{\psi}_k)_g(\theta_1, \dots, \theta_k) = (\varphi_1(g) \cdot \theta_1, \dots, \varphi_k(g) \cdot \theta_k),$$

with $(\theta_1, \dots, \theta_k) \in \mathbb{T}^k$.

Next, we introduce the action $\psi_1\colon G \times M_1 \to M_1$ of G on the manifold $M_1 = M \times T^*(\mathbb{T}^k)$ given by

$$\psi_1(g, (x, \theta, r)) = (\psi_g(x), (\bar{\psi}_k)_g(\theta), r) \quad \text{with } x \in M \text{ and } (\theta, r) \in T^*\mathbb{T}^k.$$

Then, we have that the 1-forms α_1 and ω_1 on M_1 given in (20) are G-invariant with respect to ψ_1. Moreover, if $F\colon M\times\mathbb{R}^k\to M_1$ is the embedding described in (21), the submanifold $C_1=F(M\times\mathbb{R}^k)$ is also G-invariant. In addition, under the identification of M with the reduction of M_1 by C_1, the induced action from ψ_1 on this reduced space is just ψ.

Take $M_2=\mathbb{R}\times\mathcal{J}^1M_1$ and the cotangent lift of ψ_1, $T^*\psi_1\colon G\times T^*M_1\to T^*M_1$, and we construct the action $\psi_2\colon G\times M_2\to M_2$ on M_2 given by

$$(\psi_2)_g((s,u,\alpha_{x_1}))=(s,u,T^*(\psi_1)_{g^{-1}}(\alpha_{x_1}))$$

for $(s,u,\alpha_{x_1})\in\mathbb{R}\times\mathcal{J}^1M_1$ and $x_1\in M_1$. Using the fact that ω_1 is G-invariant with respect ψ_1 and $\lambda_{M_1}\in\Omega^1(M_1)$ is G-invariant with respect to $T^*\psi_1$, we deduce that the 1-forms ω_2 and α_2 on M_2 described in (22) are G-invariant with respect to ψ_2.

Note that, since the vector fields E and B and the 1-form α_1 are invariant, it follows that the submanifold C_2 is G-invariant with respect to the action ψ_2. In fact, under the identification of M_1 with the reduction of M_2 by C_2, the induced action by ψ_2 is just ψ_1.

In the third step of the proof of theorem 3, we have that the map $H\colon M_3=M_2\to M_2$ defined in (23) is a diffeomorphism. On the other hand, one may assume that the real function f_0 is also G-invariant. It is sufficient to take

$$\widetilde{f_0}=\int_G\psi_g^*(f_0)dg$$

which is G-invariant and $d\widetilde{f_0}=\omega_0$. Thus, the diffeomorphism $H\colon M_3\to M_2$ is equivariant and it induces a new action $\psi_3\colon G\times M_3\to M_3$ such that α_3,ω_3,B_3 and E_3 are G invariant.

Since G is compact and M is of finite type, from the Mostow-Palais theorem [22, 31], we deduce that there exist an integer n, an orthogonal action of G on $\mathbb{R}^n$ and an equivariant embedding $i\colon M\hookrightarrow\mathbb{R}^n$. Therefore, we have an orthogonal action of G on $\mathbb{T}^k\times\mathbb{R}^N$ with $N=n+k$

$$\bar{\psi}_{k,N}\colon G\times\mathbb{T}^k\times\mathbb{R}^N\to\mathbb{T}^k\times\mathbb{R}^N$$

given by

$$(\bar{\psi}_{k,N})_g(z,r,a)=((\bar{\psi}_k)_g(z),r,g\cdot a),$$

with $(z,r,a)\in\mathbb{T}^k\times\mathbb{R}^k\times\mathbb{R}^n$. Thus, we may consider the l.c.s. action on $(M_{k,N},\Phi_{k,N,\mu},\alpha_{k,N},\omega_\mu)$ defined by

$$(\psi_{k,N})_g(s,u,\gamma_{(z,t)})=(s,u,T^*(\bar{\psi}_{k,N})_{g^{-1}}(\gamma_{(z,t)}))$$

for $(s,u,\gamma_{z,t})\in M_{k,N}$ and $(z,t)\in\mathbb{T}^k\times\mathbb{R}^N$. Note that the 1-form $\alpha_{k,N}=du-\lambda_{\mathbb{T}^k\times\mathbb{R}^N}$ is G-invariant with respect to $\psi_{k,N}$. Moreover, since $d\theta_j$ is G-invariant with respect to $\bar{\psi}_k$ then ω_μ is G-invariant with respect to $\psi_{k,N}$.

The induced embedding $i'\colon M_1=M\times\mathbb{T}^k\times\mathbb{R}^k\to\mathbb{T}^k\times\mathbb{R}^N$ by $i\colon M\hookrightarrow\mathbb{R}^n$, is G-invariant with respect to ψ_1 and $\bar{\psi}_{k,N}$. Thus, $i'(M_1)$ is G-invariant with respect to $\bar{\psi}_{k,N}$. Since the projection $\pi\colon M_{k,N}\to\mathbb{T}^k\times\mathbb{R}^N$ is G-invariant with respect to $\psi_{k,N}$ and $\bar{\psi}_{k,N}$ we conclude that $C_4=\pi^{-1}(i'(M_1))$ is G-invariant with respect to $\psi_{k,N}$. Finally, under the identification of M_3 with the reduction of $M_{k,N}$ by C_4, the induced action from $\psi_{k,N}$ is just ψ_3.

The action of $SL(k,\mathbb{Z})$ is equivariant w.r.t. to the action of $\bar{\psi}_{k,N}$, so may consider the $SL(k,\mathbb{Z})$-orbit and cut down the dependence of the construction from the basis μ to the lattice Λ. $\square$

Remark 14. It is natural to try to define universal models for Vaisman manifolds via reduction. One obstacle we find is that our universal l.c.s. manifolds of the first kind $(M_{k,N},\Phi_{k,N,\mu},\,\alpha_{k,N})$ do not seem to admit compatible Vaisman structures in a straightforward manner. As for the process of reduction itself, l.c.K. coisotropic

reduction can be defined in an obvious way: to define l.c.s. reduction one requires a regular foliation with smooth leaf space which integrates $\ker\Phi_{|C}$, and requires $\omega_{|C}$ to be $\mathcal{F}$-basic. The additional ingredient is an integrable compatible almost complex structure in the leaf space. It is reasonable then to further ask (1) $TC \cap JTC$ to be of constant rank and complementary to $\mathcal{F}$ in C, and (2) the CR structure $(C, TC \cap JTC)$ to be $\mathcal{F}$-basic, i.e. invariant by flows of vector fields tangent to $\mathcal{F}$. Of course, what is difficult is to give geometric conditions which imply that l.c.K. coisotropic reduction is possible. This was done in [9] for twisted Hamiltonian actions by automorphisms of the structure (preserving J and the conformal class of Φ). For Vaisman coisotropic reduction, one further adds the requirement of C being stable under the holomorphic flow of $B - iJE$. If a Vaisman manifold is acted upon by a group of Vaisman automorphisms, then the action by definition commutes with the flow of $B - iJE$, it is twisted Hamiltonian, and free on the inverse image of zero if this is non-empty [9], so coisotropic Vaisman reduction is possible.

8. Conclusions and future work

Universal models for several types of l.c.s. manifolds associated with embedding or reduction procedures are obtained. The existence of these universal models for embeddings (in the compact case) is related with the search of a universal model for a compact manifold M endowed with an arbitrary 1-form Θ. In this case one may embed the manifold into a sphere S^{2N-1} and the pullback of the standard contact 1-form on S^{2N-1} is just Θ (up to the multiplication by a positive constant). In relation with previous results, our method allows to cut down substantially the dimension of the sphere. In the particular case of a compact contact manifold M, we give a simple proof about how to obtain a contact embedding (up to the multiplication by a positive constant) from M to S^{2N-1}.

Using these results, we have seen that the universal model (via embeddings) of a compact exact l.c.s. manifold with integral period lattice is the cartesian product $S^{2N-1} \times S^1$ with the standard l.c.s. structure. In the particular case of a l.c.K. structure with automorphic potential and integral period lattice on a compact manifold M, we have discussed the relation between the l.c.s. embedding of M into $S^{2N-1} \times S^1$ and recent holomorphic embedding results for this type of manifolds.

Finally, we have obtained that a universal model for a l.c.s. manifold (of finite type) of the first kind via a reduction procedure is $\mathbb{R} \times \mathcal{J}^1(\mathbb{T}^k \times \mathbb{R}^N)$ endowed with a suitable l.c.s. structure. An equivariant version of this result has been presented at the end of the paper.

It would be interesting to pursue the existence of universal models (for embedding and reduction procedures) for arbitrary l.c.s. manifolds.

Appendix A. Non-exactness of the Oeljeklaus-Toma l.c.K. structures

In this appendix we will show that the Oeljeklaus-Toma l.c.K. manifolds are not exact.

We briefly recall the construction of the Oeljeklaus-Toma l.c.K. structures (for details see [24, 32] and references therein):

Let K be an algebraic number field of degree n and let $\sigma_1, \ldots, \sigma_n$ be the distinct embeddings of K into $\mathbb{C}$. Assume that $\sigma_1, \ldots, \sigma_{n-2}$ are real and σ_{n-1} and σ_n are non-real. Let $\mathcal{O}_K$ denote the ring of algebraic integers of K, which is a rank n free $\mathbb{Z}$-module. Let $\mathcal{O}_K^{*,+}$ denote the positive units, i.e. those units $u \in \mathcal{O}_K^*$ such that

$$\sigma_i(u) > 0, \, i = 1, \ldots, n-2.$$

According to Oeljeklaus and Toma the actions

$$T_a(z_1,\dots,z_{n-1}) := (z_1+\sigma_1(a),\dots,z_{n-1}+\sigma_{n-1}(a)),\ a\in\mathcal{O}_K,$$

$$R_u(z_1,\dots,z_{n-1}) := (\sigma_1(u)z_1,\dots,\sigma_{n-1}(u)z_{n-1}),\ u\in\mathcal{O}_K^{*,+}$$

fit into a free co-compact action of the semi-direct product $\mathcal{O}_K\rtimes\mathcal{O}_K^{*,+}$ on $\mathbb{H}^{n-2}\times\mathbb{C}$, where $\mathbb{H}$ denotes the upper half plane. The corresponding quotient

$$(M_K,J_K) := \mathbb{H}^{n-2}\times\mathbb{C}/\mathcal{O}_K\rtimes\mathcal{O}_K^{*,+}$$

is called an Oeljeklaus-Toma manifold.

Consider the function

$$\begin{array}{rcl} r\colon \mathbb{H}^{n-2} & \longrightarrow & \mathbb{R} \\ (z_1,\dots,z_{n-2}) & \longmapsto & \prod_{i=1}^{n-2}(\mathrm{im} z_i)^{-1} \end{array} \tag{24}$$

and the standard 2-form

$$\Phi_{\mathrm{std}} = dz_{n-1}\wedge d\bar{z}_{n-1}\in\Omega^{1,1}(\mathbb{C}),$$

and define

$$\Omega = \partial\bar{\partial}\phi + \Phi_{\mathrm{std}}.$$

Then Ω is a Kähler form on $\mathbb{H}^{n-2}\times\mathbb{C}$ such that

$$T_a^*\Omega = \Omega,\ a\in\mathcal{O}_K, \tag{25}$$

and

$$R_u^*\Omega = |\sigma_{n-1}(u)|^2\Omega,\ u\in\mathcal{O}_K^{*,+}.$$

Hence $\mathcal{O}_K\rtimes\mathcal{O}_K^{*,+}$ acts by Kähler homotheties giving rise to a multiplicative character χ, and therefore the Kähler form descends to a conformal class of l.c.K. structures with Lee class associated to χ (see lemma 1). We let Φ_K be a representative of the induced conformal class of l.c.K. structures.

Note that r in (24) is an automorphic function, but the function $r+z_{n-1}\bar{z}_{n-1}$ -which is a dd^c-potential for Ω- is not automorphic. That no automorphic potential for Ω can exist is a consequence of the following result:

Proposition 3. *The Oeljeklaus-Toma l.c.K. manifold (M_K,J_K,Φ_K) is non-exact.*

Proof. By lemma 1 exactness of (M_K,J_K,Φ_K) is equivalent to

$$\Omega = d\alpha,\ \alpha\in\Omega^1(\mathbb{H}^{n-2}\times\mathbb{C})^\chi. \tag{26}$$

Because r is automorphic (26) is equivalent to

$$\Phi_{\mathrm{std}} = d\alpha,\ \alpha\in\Omega^1(\mathbb{H}^{n-2}\times\mathbb{C})^\chi. \tag{27}$$

Let us assume that (27) holds.

Let us write $\mathbb{H}=\mathbb{R}\times\mathbb{R}^{>0}$ and

$$\mathbb{H}^{n-2}\times\mathbb{C} = (\mathbb{R}^{>0})^{n-2}\times\mathbb{R}^{n-2}\times\mathbb{C}.$$

Because (i) the action by translations of $\mathcal{O}_K$ on $\mathbb{H}^{n-2}\times\mathbb{C}$ is trivial on the factor $(\mathbb{R}^{>0})^{n-2}$ ($\sigma_i(a)\in\mathbb{R}$, $a\in\mathcal{O}_K$, $i=1,\dots,n-2$) and (ii) $\sigma(\mathcal{O}_K)\subset\mathbb{R}^{n-2}\times\mathbb{C}$ is a lattice of full rank [24], we have

$$\mathbb{H}^{n-2}\times\mathbb{C}/\mathcal{O}_K\cong(\mathbb{R}^{>0})^{n-2}\times\mathbb{T}^n.$$

According to (25) the restriction of χ to $\mathcal{O}_K$ is trivial and thus both α and Φ_{std} descend to forms $\hat{\alpha},\hat{\Phi}_{\mathrm{std}}$ on $(\mathbb{R}^{>0})^{n-2}\times\mathbb{T}^n$. By (27) we obtain

$$\hat{\Phi}_{\mathrm{std}} = d\hat{\alpha}. \tag{28}$$

Note that because Φ_{std} is constant, it is invariant by any translation in $(\mathbb{R}^{>0})^{n-2} \times \mathbb{R}^{n-2} \times \mathbb{C}$. In particular $\hat{\Phi}_{\mathrm{std}} \in \Omega^2((\mathbb{R}^{>0})^{n-2} \times \mathbb{T}^n)$ is invariant by the $\mathbb{T}^n$-action. Fix a Haar measure in $\mathbb{T}^n$ of total volume 1 and denote the average of $\hat{\alpha}$ by $\int \hat{\alpha}$. Average (28) and use the invariance of $\hat{\Phi}_{\mathrm{std}}$ to get

$$\hat{\Phi}_{\mathrm{std}} = d \int \hat{\alpha}. \tag{29}$$

Fix any $\{y\} \in (\mathbb{R}^{>0})^{n-2}$ and the corresponding torus $\mathbb{T}^n := \{y\} \times \mathbb{T}^n$. The result of restricting (29) to this torus is

$$\hat{\Phi}_{\mathrm{std}|\mathbb{T}^n} = d(\int \hat{\alpha}_{|\mathbb{T}^n}). \tag{30}$$

Observe that by construction $\hat{\Phi}_{\mathrm{std}|\mathbb{T}^n}$ is a non-trivial 2-form. On the other hand the restriction $\int \hat{\alpha}_{|\mathbb{T}^n}$ is an invariant 1-form, and thus its exterior differential must vanish, which contradicts (30).

□

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Juan C. Marrero: ULL-CSIC Geometría Diferencial y Mecánica Geométrica, Departamento de Matemática Fundamental, Facultad de Matemáticas, Universidad de la Laguna, La Laguna, Tenerife, Canary Islands, Spain

E-mail address: jcmarrer@ull.es

David Martínez: Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

E-mail address: martinez@math.ist.utl.pt

Edith Padrón: ULL-CSIC Geometría Diferencial y Mecánica Geométrica, Departamento de Matemática Fundamental, Facultad de Matemáticas, Universidad de la Laguna, La Laguna, Tenerife, Canary Islands, Spain

E-mail address: mepadron@ull.es